Pion weak decay constant at finite density
from the instanton vacuum

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Abstract

We investigate the pion weak decay constant (F_π) and pion mass (m_π) at finite density within the framework of the nonlocal chiral quark model from the instanton vacuum with the finite quark-number chemical potential (μ) taken into account. We mainly focus on the Nambu-Goldstone phase below the critical value of the chemical potential $\mu_c \approx 320$ MeV, which is determined consistently within the present framework. The breakdown of Lorentz invariance at finite density being considered, the time (F_π^t) and space (F_π^s) components are computed separately, and the corresponding results turn out to be: $F_\pi^t = 82.96$ MeV and $F_\pi^s = 80.29$ MeV at μ_c, respectively. Using the in-medium Gell-Mann-Oakes-Renner (GOR) relation, we show that the pion mass increases by about 15% at μ_c.

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I. INTRODUCTION

The in-medium modifications of the pion have been one of the most interesting issues both in experimental and theoretical hadron physics. The pion is identified as the Goldstone boson arising from the spontaneous breakdown of chiral symmetry (SBχS) which is essential in describing low-energy hadronic phenomena. Since chiral symmetry is expected to be restored at high temperature and density, the changes of pion properties in medium will provide crucial information on the restoration of chiral symmetry. Among the properties of the pion, its weak decay constant (F_π) and mass (m_π) are the most important quantities, since they are deeply related to the SBχS: The F_π and m_π are related to the quark mass and chiral condensate via the Gell-Mann-Oakes-Renner (GOR) relation [1], so that the in-medium modification of the F_π and m_π will give a key clue for understanding the mechanism of the chiral symmetry restoration in matter.

Experimentally, the modifications of the F_π and m_π can be measured from deeply bound pionic atoms, the s-wave pion-nucleus interaction being considered in medium [2, 3, 4, 5, 6], as suggested by Refs. [7, 8, 9] and other related works [10, 11]. The in-medium change of the pion mass can be also probed in the Drell-Yan process [12, 13].

There has been a great deal of theoretical work on the in-medium modifications of the F_π and m_π. For example, meson-baryon chiral perturbation theory (χPT) and models with chiral symmetry were applied for this purpose [14, 15, 16, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. Since the Lorentz invariance is broken in medium, one has to study the space and time components of the pion weak decay constant separately. In in-medium χPT [15, 17] for instance, the magnitude of its space component F_π^s was shown to be about four times smaller than that of the time component F_π^t at normal nuclear density $\rho_0 \approx 0.17\,\mathrm{fm}^{-3}$. Moreover, the chiral condensate varies sizably in medium [16, 17, 23, 24, 25, 26]. It was also shown that within these approaches, the contribution of the intermediate Δ isobar explains the suppression of the F_π^s in comparison with the F_π^t. In the QCD sum rules, it was discussed that dimension-five operators are responsible for making splitting between F_π^t and F_π^s, and the contribution of the intermediate Δ isobar contribution makes F_π^s much smaller than F_π^t, while they are approximately in the same order without it [14]. Taking into account the experiments conducted in Ref. [6, 27] and using effective potential models, Refs. [28, 29] have studied the deeply bound pionic atoms and have shown that the encoded pion-mass modification turned out to be also small (about 10%).

In the present work, we would like to investigate the modifications of the F_π and m_π at finite quark-number chemical potential ($\mu \neq 0$) but at zero temperature ($T = 0$), employing the nonlocal chiral quark model (NLχQM) that is derived from the nontrivial instanton vacuum [30] with the finite quark-number chemical potential considered in the N_c limit [31]. The NLχQM from the instanton vacuum is characterized by the average instanton size $\bar{\rho} \approx 1/3$ fm and inter-instanton distance $\bar{R} \approx 1$ fm. The scale of the model is given by the average instanton size, $i.e.$ $\Lambda \approx 1/\bar{\rho} \approx 600$ MeV. We already have successfully applied this modified NLχQM to the pion electromagnetic form factor [32] and magnetic susceptibility of the QCD vacuum [33]. As done previously, we will mainly focus on the Nambu-Goldstone (NG) phase below $\mu = \mu_c \approx 320$ MeV, which is close to ρ_0. We will show in the present work that the time and space components of the pion weak decay constant will turn out to be $F_\pi^t = 82.96$ MeV and $F_\pi^s = 80.29$ MeV at μ_c, which are about $13 \sim 16\%$ smaller than that in free space ($F_\pi = 93$ MeV). The results are compatible with those obtained in other models, though the result for the F_π^s seems to be larger than those from χPT and from the

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QCD sum rules. Using the GOR relation that is satisfied within the model [30], we estimate the pion mass shift, resulting in about 15% increase at μ_c.

The present work is organized as follows: In Section II, we briefly review the general formalism in the NLχQM at finite density. In Section III, we discuss the phase structures in the present framework. In Section IV, the numerical results are given and discussed. The final Section is devoted to summary and conclusions.

II. NONLOCAL CHIRAL QUARK MODEL AT FINITE DENSITY

The Dirac equation in the presence of the finite quark-number chemical potential (μ) in the instanton (anti-instanton) background field can be written as follows:

$$[i\slashed{\partial} - i\gamma_4\mu - \slashed{A}_{I\bar{I}}]\,\Psi^{(n)}_{I\bar{I}} = \lambda_n \Psi^{(n)}_{I\bar{I}}. \tag{1}$$

In the present work, we work in Euclidean space and assume the chiral limit ($m_q = m_\mathrm{u} = m_\mathrm{d} = 0$). The subscript ($\bar{I}$)$I$ stands for the (anti)instanton contribution, and we use a singular-gauge instanton solution:

$$A^\alpha_{I\bar{I}\mu}(x) = \frac{2\bar{\eta}^{\alpha\nu}_\mu \bar{\rho}^2 x_\nu}{x^2(x^2 + \bar{\rho}^2)}, \tag{2}$$

where $\eta^{\alpha\nu}_\mu$ and $\bar{\rho}$ denote the 't Hooft symbol and average instanton size, respectively. The quark zero-mode solution can be obtained in the presence of the μ as follows:

$$[i\slashed{\partial} - i\gamma_4\mu - \slashed{A}_{I\bar{I}}]\,\Psi^{(0)}_{I\bar{I}} = 0. \tag{3}$$

The explicit form of $\Psi^{(0)}$ can be found in Ref. [31]. The effective chiral action can be constructed by the would-be zero mode of Eq.(3). The quark propagator in one instanton background goes to infinity in the chiral limit ($m_q \to 0$) [1]:

$$S_{I\bar{I}}(x,y) = \langle \psi(x)\psi(y)^\dagger \rangle = -\sum_n \frac{\Psi^{(n)}_{I\bar{I}}(x)\Psi^{(n)\dagger}_{I\bar{I}}}{\lambda_n + im_q} = S'_{I\bar{I}}(x,y) - \frac{\Psi^{(0)}_{I\bar{I}}(x)\Psi^{(0)\dagger}_{I\bar{I}}(y)}{im_q}, \tag{4}$$

where $S'_{I\bar{I}}$ denotes the non-zero mode contribution. Since the zero-mode contribution dominates in Eq.(4) at small momenta ($p \lesssim 1/\bar{\rho}$) whereas it is reduced to the free quark propagator at large momenta ($p \gg 1/\bar{\rho}$), the quark propagator can be approximated as

$$S_{I\bar{I}}(x,y) \approx S_0 - \frac{\Psi^{(0)}_{I\bar{I}}\Psi^{(0)\dagger}_{I\bar{I}}}{im_q}, \tag{5}$$

where S_0 is a free quark propagator defined as $(i\slashed{\partial} - i\gamma_4\mu)^{-1}$. Starting from the zero-mode approximation in Eq.(5), one can derive the quark propagator in the instanton ensemble [30, 31]:

$$S = \frac{1}{i\slashed{\partial} - i\gamma_4\mu + iM(i\partial, \mu)}, \tag{6}$$

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[1] Note that the chiral limit will be taken after the low-energy effective partition function is obtained.

where M denotes the momentum-dependent and μ-dependent quark mass that arises from the Fourier transform of the quark zero-mode solution:

$$M(\bar{k}) = M_0(\mu)\bar{k}^2\psi^2(\bar{k}). \tag{7}$$

Here, $\bar{k} = (\vec{k}, k_4 + i\mu)$. The M_0 is the constituent quark mass at $k^2 = 0$, which depends on μ. It will be determined consistently within the model. The analytical expressions for ψ_4 and $\vec{\psi}$ are given in Appendix [31].

The low-energy effective partition function of the NLχQM with $\mu \neq 0$ can be written as follows:

$$\mathcal{Z}_{\text{eff}} = \int d\lambda \, D\psi \, D\psi^\dagger \exp\left[\int d^4x\psi^\dagger(i\slashed{\partial} - i\gamma_4\mu)\psi + \lambda(Y^+ + Y^-) + N\left(\ln\frac{N}{\lambda V\mathcal{M}} - 1\right)\right], \tag{8}$$

where V indicates the four-dimensional volume, whereas N represents the average number of the (anti)instantons, $N = (N_+ + N_-)/2$. The variational parameter λ plays a role of a Lagrangian multiplier. The $Y^\pm$ stands for the $2N_f$-'t Hooft interaction in the instanton background with nonzero μ. The parameter $\mathcal{M}$ is required to make the argument of the logarithm dimensionless. All calculations are performed to order $\mathcal{O}(\lambda)$.

III. PHASE STRUCTURES: NG AND CSC PHASES

In this Section, we want to discuss the phase structure for $\mu \neq 0$ and $T = 0$. Since we are interested in the case of $N_f = 2$ and $N_c = 3$, there are various phase structures characterized by the different order parameters g and f for the NG and CSC phases, respectively [31]. They can be computed from the quark loops of the normal (G) and abnormal (F) quark propagators, which correspond to the Dyson-Schwinger-Gorkov (DSG) equations:

$$
\begin{aligned}
Z(k) &= 1 - G(k)A(k,\mu)M_0, \\
G(k) &= Z(k)\psi^2(p)M_0, \\
F(k) &= 2Z(-k)\psi_\mu(k,\mu)\psi^\mu(-k,\mu)\Delta,
\end{aligned}
\tag{9}
$$

where the vertex functions are defined by

$$
\begin{aligned}
A(k,\mu) &= (k+i\mu)^2\psi^2(k,\mu), \\
B(k,\mu) &= (k^2+\mu^2)\psi_\mu(k,\mu)\psi^\mu(-k,\mu) + (k+i\mu)_\mu\psi^\mu(k,\mu)(k-i\mu)_\nu\psi^\nu(-k,\mu) \\
&\quad - (k+i\mu)_\mu\psi^\mu(-k,\mu)(k-i\mu)_\nu\psi^\nu(k,\mu).
\end{aligned}
\tag{10}
$$

Here, Δ stands for the diquark energy gap, corresponding to the diquark correlation. Note that we consider only the pure NG and CSC phases here for simplicity. Hence, the metastable mixed phases of the NG and CSC are not taken into account. The two phases are then characterized by $g \neq 0$ and $f = 0$ for the NG phase and vice versa for the CSC one. Using Eqs. (9) and (10), the condensates f and g can be written as follows:

$$g(\mu) = \frac{\lambda M_0}{N_c^2 - 1}\int\frac{d^4k}{(2\pi)^4}\frac{\alpha(k,\mu)}{1+\alpha(k,\mu)M_0^2}, \quad f(\mu) = \frac{2\lambda\Delta}{N_c^2 - 1}\int\frac{d^4k}{(2\pi)^4}\frac{\beta(k,\mu)}{1+4\beta(k,\mu)\Delta^2}, \tag{11}$$

where

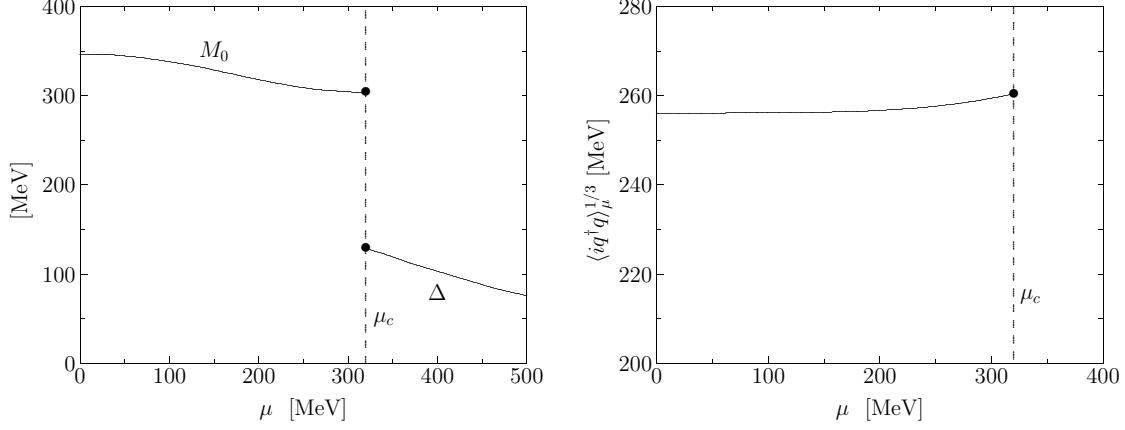$$\alpha(k,\mu) = A(k,\mu)\psi^2(k,\mu), \quad \beta(k,\mu) = B(k,\mu)\psi_\mu(k,\mu)\psi^\mu(-k,\mu). \tag{12}$$

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FIG. 1: M_0 and Δ (left), and $\langle iq^\dagger q \rangle^{\frac{1}{3}}$ (right) as functions of the quark-number chemical potential μ up to $\mu = \mu_c \approx 320$ MeV. Vertical dashed lines indicate the critical density, $\mu_c \approx 320$ MeV.

In turn, M_0 and Δ can be also expressed in terms of g and f:

$$M_0 = \left(2N_c - \frac{2}{N_c} \right) g(\mu), \quad \Delta = \left(1 + \frac{1}{N_c} \right) f(\mu). \tag{13}$$

Differentiating the partition function of Eq. (8) with respect to λ, we can obtain the following saddle-point equation:

$$\frac{N}{V} = \lambda \langle Y^+ + Y^- \rangle = \frac{4(N_c^2 - 1)}{\lambda} \int \frac{d^4 k}{(2\pi)^4} \left[N_c M_0 G(k) + 4\Delta F(k) \right]. \tag{14}$$

Since $\langle Y^+ + Y^- \rangle$ corresponds to a ∞-shape quark-loop integral with G and F for $N_f = 2$, it can be rewritten in terms of f and g as given above. Note that there is one caveat: We assume that there is no density dependence in the instanton packing fraction N/V for the $\mathcal{O}(\lambda)$. Thus, we use $N/V \approx (200 \, \mathrm{MeV})^4$ for $\mu \geq 0$. Inserting Eq. (11) into Eq. (14) for the NG and CSC phase regions, we can obtain M_0 and Δ numerically from the saddle-point equation. In the left panel of Fig. 1, we draw M_0 and Δ as a function of μ. The critical density is determined by the following condition [31]:

$$\left. \frac{f(\mu)}{g(\mu)} \right|_{\mu = \mu_c} = \left[\frac{N_c(N_c - 1)}{2} \right]^{\frac{1}{2}}, \tag{15}$$

where μ_c becomes about 320 MeV for $\bar{R} \approx 1$ fm and $\bar{\rho} = \frac{1}{3}$ fm. In Fig 1, μ_c is indicated by vertical dashed lines. Note that M_0 decreases as μ increases and disappears for the region beyond μ_c. The diquark energy gap starts to exist ($\Delta \approx 120$ MeV) above the critical point. These numerical results are basically obtained in Ref. [31].

Similarly, we can also derive the chiral condensate from the partition function, which is another order parameter for the NG phase:

$$\langle iq^\dagger q \rangle_\mu = 4N_c \int \frac{d^4 k}{(2\pi)^4} \left[\frac{M(k, \mu)}{(k + i\mu)^2 + M^2(k, \mu)} \right]. \tag{16}$$

In the right panel of Fig. 1, we show the numerical results for the chiral condensate with respect to the μ. It turns out that it increase slowly as μ does. At the critical point, it

becomes larger than its vacuum value by about $4 \sim 5\%$. Thus, we can conclude that the chiral condensate remains nearly the same as that for free space within the NG phase, as pointed out in Refs. [35, 36]. A similar tendency was also shown in the Nambu-Jona-Lasinio model (NJL) [37, 38] and in the two-color lattice simulation [39]. However, note that the present result is rather different from those computed in in-medium χPT and other effective model calcultaions [16, 17, 23, 24, 25, 26] in which it was shown that the condensate decreases as nuclear density increases.

IV. PION WEAK DECAY CONSTANT IN MEDIUM

The pion weak decay constant F_π can be defined as the following transition matrix element:

$$\langle 0|A_\mu^a(x)|\pi^b(P)\rangle = i\sqrt{2}F_\pi \delta^{ab} P_\mu e^{-iP\cdot x}, \tag{17}$$

where A_μ^a and P_μ denote the axial-vector current, $\psi^\dagger \gamma_5 \gamma_\mu \frac{\tau^a}{2}\psi$, and the pion on-shell momentum $P^2 = m_\pi^2$, respectively. Since the Lorentz invariance of the matrix element is broken at finite density, Eq. (17) should be decomposed into the space and time parts as follows:

$$\langle 0|A_i^a(x)|\pi^b(P)\rangle = i\sqrt{2}F_\pi^s \delta^{ab} P_i e^{-iP\cdot x}, \quad \langle 0|A_4^a(x)|\pi^b(P)\rangle = i\sqrt{2}F_\pi^t \delta^{ab} P_4 e^{-iP\cdot x}. \tag{18}$$

From the low-energy effective partition function given in Eq. (8), one can derive an effective chiral action in terms of the quarks and NG boson fields π^a with the bosonization carried out [34]:

$$\mathcal{S}_{\text{eff}}[\pi, \mu] = -\text{Sp}\ln\left[i\bar{\partial}\!\!\!/ + i\sqrt{M(i\bar{\partial})}U_5\sqrt{M(i\bar{\partial})}\right], \tag{19}$$

where Sp indicates the functional trace over color (c), flavor (f) and Dirac (γ) spaces, $i.e.$ $\int d^4 x\, \text{Tr}_{c,f,\gamma}\langle\cdots\rangle$. The density-modified covariant derivative is defined as $i\bar{\partial}_\mu = i\partial_\mu - i(\vec{0}, \mu)$. The U_5 stands for the nonlinear NG boson field:

$$U_5 = \frac{1+\gamma_5}{2}U + \frac{1-\gamma_5}{2}U^\dagger = \exp\left(\frac{i\gamma_5 \boldsymbol{\pi}\cdot\boldsymbol{\tau}}{F_0}\right). \tag{20}$$

Here F_0 and τ^a denote a generic normalization constant for π and the Pauli matrix, respectively. We use the following expression for π^a:

$$\boldsymbol{\pi}\cdot\boldsymbol{\tau} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & \pi^0 \end{pmatrix}. \tag{21}$$

We would like to emphasize that the pion field π^a in Eq. (19) is not a physical one, since it is introduced as an auxiliary field in the bosonization. Considering the field renormalization, we can write π_{phy}^a as

$$\pi_{\text{phy}}^a = \frac{1}{C_r}\pi^a, \tag{22}$$

where C_r stands for a certain field-renormalization parameter. In the presence of the quark-number chemical potential, the physical NG-boson field itself can be modified as the Migdal field in meson-nucleon χPT with nonzero baryon density ρ [40]. Thus, C_r is given in principle

as a function of μ, *i.e.* $C_r(\mu)$, which depends on an appropriate renormalization scale. We, however, have assumed that the generic normalization constant F_0 does not change at finite density.

In order to compute the pion weak decay constant in Eq. (17), we rewrite the effective chiral action in the presence of an external axial-vector source $J_{5\mu}^a$:

$$\mathcal{S}_{\text{eff}}[\pi, \mu, J_{5\mu}^a] = -\text{Sp} \ln \left[i\bar{\partial} + \gamma_5 \gamma^\mu \frac{\tau^a}{2} J_{5\mu}^a + \sqrt{M(i\bar{\partial}, J_{5\mu}^a)} U_5 \sqrt{M(i\bar{\partial}, J_{5\mu}^a)} \right]. \tag{23}$$

Since it is well known that the gauge invariance is broken in the presence of a nonlocal interaction, we have to make the effective chiral action gauge invariant. In fact, this gauge-invariant problem for the nonlocal chiral quark model from the instanton vacuum has been already treated in Refs. [41, 42, 43] to which we refer for details. When the external gauge field is weak, we can simply replace the usual derivatives by the covariant ones for soft external vector and axial-vector fields. Thus, note that the modified derivative inside $\sqrt{M}$ has been also replaced by $i\bar{\partial} + \gamma_5 \gamma^\mu \frac{\tau^a}{2} J_{5\mu}^a$.

Using the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula and Eq. (17), one can obtain the following expression [44]:

$$i\sqrt{2}\delta^{ab} F_\pi(q^2, \mu) q_\mu = \mathcal{K}_\pi \int d^4x \langle 0|T[A_\mu^a(x)\pi_{\text{phy}}^b(0)]|0\rangle e^{iq\cdot x}$$
$$= \frac{\mathcal{K}_\pi}{C_r(\mu)} \int d^4x \langle 0|T[A_\mu^a(x)\pi^b(0)]|0\rangle e^{iq\cdot x}, \tag{24}$$

where $\mathcal{K}_\pi$ denotes the inverse of the pion propagator, $\mathcal{K}_\pi = q^2 + m_\pi^{*2}$, in which the asterisk designates the density modification. The physical pion weak decay constant is defined at $q^2 \to -m_\pi^{*2}$. Eq. (24) can be further evaluated by the second functional derivative of Eq. (23) with respect to the $J_{5\mu}^a$ and source field J_5^b for the pion:

$$\langle 0|T[A_\mu^a(x)\pi^b(0)]|0\rangle = \frac{\delta^2 \ln \mathcal{Z}_{\text{eff}}[\pi, \mu, J_{5\mu}^a]}{\delta J_{5\mu}^a(x)\,\delta J_5^b(0)} = \int d^4z \frac{\delta^2 \mathcal{S}_{\text{eff}}[\pi, \mu, J_{5\mu}^a]}{\delta J_{5\mu}^a(x)\,\delta \pi^b(z)} \mathcal{K}_\pi^{-1}(z). \tag{25}$$

Having performed some tedious calculation, we arrive at

$$F_\pi(\mu) P_\mu = \frac{4N_c}{C_r F_0} \int \frac{d^4k}{(2\pi)^4} \left[\underbrace{\frac{\sqrt{M(\bar{k})M(\bar{k}-P)}[(P_\mu - \bar{k}_\mu)M(\bar{k}) + \bar{k}_\mu M(\bar{k}-P)]}{[\bar{k}^2 + M^2(\bar{k})][(\bar{k}-P)^2 + M^2(\bar{k}-P)]}}_{\text{local cont.}} \right.$$
$$\left. - \underbrace{\frac{M(\bar{k})\sqrt{M(\bar{k})}\sqrt{M(\bar{k}-P)}_\mu - M(\bar{k})\sqrt{M(\bar{k})}_\mu \sqrt{M(\bar{k}-P)}}{\bar{k}^2 + M^2(\bar{k})}}_{\text{nonlocal cont.}} \right], \tag{26}$$

where $\bar{k} = (\vec{k}, k_4 + i\mu)$ and $C_r = C_r(\mu)$. We also have used the following expression:

$$\sqrt{M(k)}_\mu = \frac{\partial \sqrt{M(k)}}{\partial k_\mu}. \tag{27}$$

Note that F_π in Eq. (26) consists of two contributions, that is, the local (L) and nonlocal (NL) ones. The later contains derivatives of $M(\bar{k})$, while the first not. Using Eq. (26) and

relevant expansions with respect to the momentum given in Appendix, we can compute F_π^s and F_π^t separately. The local contributions are obtained as

$$F_{\pi,\mathrm{L}}^s(\mu) = \frac{4N_c}{C_r F_0} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+\mathcal{M}^2)^2} \left[\mathcal{M}^2 - \frac{1}{2}k^2\mathcal{M}\tilde{\mathcal{M}}' - 5\mu^2\,k_4^2\tilde{\mathcal{M}}'^2 \right], \tag{28}$$

$$F_{\pi,\mathrm{L}}^t(\mu) = \frac{4N_c}{C_r F_0} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+\mathcal{M}^2)^2} \left[\mathcal{M}^2 - \frac{1}{2}k^2\mathcal{M}\tilde{\mathcal{M}}' - \mu^2\,k_4^2\tilde{\mathcal{M}}'^2 \right], \tag{29}$$

where $\mathcal{M} = \mathcal{M}(k)$ is the momentum-dependent quark mass in free space (see Appendix). In deriving Eqs. (28) and (29), we have assumed the soft pion, i.e. $P_4 \ll \frac{1}{\rho} \approx 600$ MeV and $P^2 = m_\pi^2 = 0$. From Eq. (29), one can easily see that the time component must be larger than the space one for the local contribution. Similarly, the nonlocal contributions can be evaluated as follows:

$$F_{\pi,\mathrm{NL}}^s(\mu) = -\frac{4N_c}{C_r F_0} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+\mathcal{M}^2} \left[\mathcal{M}\tilde{\mathcal{M}}' + \frac{1}{2}k^2\mathcal{M}\tilde{\mathcal{M}}'' - \frac{1}{2}k^2\tilde{\mathcal{M}}'^2 - 4\mu^2 k_4^2\tilde{\mathcal{M}}'\tilde{\mathcal{M}}'' \right], \tag{30}$$

$$F_{\pi,\mathrm{NL}}^t(\mu) = F_{\pi,\mathrm{NL}}^s(\mu). \tag{31}$$

While the time and space components are different each other for the local contribution, they turn out to be the same for the nonlocal one at the leading order because of the soft pion.

When μ is switched off, the time component equals the space one, i.e. $F_\pi^s = F_\pi^t$ as expected, and the analytic expression for $F_\pi(0)$ leads to the following:

$$F_\pi(0) = \frac{4N_c}{C_r F_0} \int \frac{d^4k}{(2\pi)^4} \left[\frac{\mathcal{M}^2 - \frac{1}{2}k^2\mathcal{M}\tilde{\mathcal{M}}'}{(k^2+\mathcal{M}^2)^2} - \frac{\mathcal{M}\tilde{\mathcal{M}}' + \frac{1}{2}k^2\mathcal{M}\tilde{\mathcal{M}}'' - \frac{1}{2}k^2\tilde{\mathcal{M}}'^2}{k^2+\mathcal{M}^2} \right], \tag{32}$$

which is already obtained in several works, for example, in Refs. [45, 46]. The $C_r F_0$ can be determined within the present framework, resulting in $C_r F_0 \approx F_\pi^{\mathrm{exp}} \approx 93$ MeV [47].

The quark-number chemical potential μ being turned on, the field-renormalization constant C_r may in principle have some modifications due to medium effects. Hence, we modify Eq. (26) simply, replacing it by $C_r(\mu)F_0 \to F_\pi(\mu)$ in the denominator. Finally, we arrive at the following expressions for the F_π^s and F_π^t from Eq. (26):

$$F_\pi^2(\mu)P_\mu = \left([F_\pi^s(\mu)]^2 P_i, [F_\pi^t(\mu)]^2 P_4 \right), \tag{33}$$

where

$$[F_\pi^s(\mu)]^2 = [F_\pi^{\mathrm{exp}}]^2 + 4N_c\,\mu^2 \int \frac{d^4k}{(2\pi)^4} \left(\frac{4k_4^2\tilde{\mathcal{M}}'\tilde{\mathcal{M}}''}{k^2+\mathcal{M}^2} - \frac{5k_4^2\tilde{\mathcal{M}}'^2}{[k^2+\mathcal{M}^2]^2} \right),$$

$$[F_\pi^t(\mu)]^2 = [F_\pi^s(\mu)]^2 + 4N_c\,\mu^2 \int \frac{d^4k}{(2\pi)^4} \frac{4k_4^2\tilde{\mathcal{M}}'^2}{[k^2+\mathcal{M}^2]^2}, \tag{34}$$

which we will solve numerically in the next Section.

V. NUMERICAL RESULTS

We now present numerical results for the pion weak decay constant at finite density. In the left panel of Fig. 2, we show F_π^s and F_π^t as functions of μ. As shown in Fig. 2, the time component of F_π is larger than that of the space one, whereas both of them decrease smoothly with respect to μ. At $\mu = 0$ we find $F_\pi^s = F_\pi^t \approx 93$ MeV as it should be, and we obtain $F_\pi^t \approx 82.96$ MeV and $F_\pi^s \approx 80.29$ MeV at $\mu = \mu_c \approx 320$ MeV. When we examine the ratio $F^{(s,t)}/F_\pi^{\mathrm{exp}}$, it must be unity at $\mu = 0$ and then it is getting smaller gradually as μ increases. At the critical density (μ_c), it turns out that $F_\pi^t/F_\pi^{\mathrm{exp}} \approx 0.89$ and $F_\pi^s/F_\pi^{\mathrm{exp}} \approx 0.86$. From these observations, F_π^s/F_π^t is less than unity for the whole region of the NG phase, and the F_π is reduced by about $13 \sim 16\%$. We summarize the results in Table I.

We are now in a position to discuss our results in comparison to those from other theoretical approaches. In the QCD sum rule [14], it was discussed that the splitting between the time and space components is represented by the dimension-five condensates at finite density. However, the computed ratios are smaller than ours by about $10\,(20)\%$: $F_\pi^t/F_\pi^{\mathrm{exp}} \approx 0.79\,(0.69)$ and $F_\pi^s/F_\pi^{\mathrm{exp}} \approx 0.78\,(0.68)$ for $m_\pi = 139\,(159)$ MeV at normal nuclear matter density $\rho_0 = 0.17\,\mathrm{fm}^{-3}$ corresponding approximately to $\mu \approx 300\,\mathrm{MeV}$ [35]. Especially, $F_\pi^s/F_\pi^{\mathrm{exp}}$ becomes much smaller, when the intermediate Δ state is considered ($0.78 \rightarrow 0.57$). The ratios were also studied in in-medium χPT in the heavy baryon limit [15, 17]: It was found that $F_\pi^t/F_\pi^{\mathrm{exp}} \approx 0.90$, which is compatible with the present results, whereas the space component turns out to be much smaller: $F_\pi^s/F_\pi^{\mathrm{exp}} \approx 0.25$ at ρ_0.

We would like to discuss more on the splitting between F_π^s and F_π^t, since we have not observed such a large difference between F_π^s and F_π^t in the present framework. For instance, in Ref. [15], the analytic expressions for F_π at finite density are given from the axial-vector and pseudoscalar correlators, being developed at the tree level as follows:

$$
\begin{aligned}
F_\pi^s(\rho_0) &= \left[1 + \frac{2c_3\rho_0}{(F_\pi^{\mathrm{exp}})^2}\right]\left[1 - \frac{\Sigma_{\pi N}\,\rho_0}{(F_\pi^{\mathrm{exp}})^2 m_\pi^2}\right]^{-1}, \\
F_\pi^t(\rho_0) &= \left[1 + \frac{2(c_2+c_3)\rho_0}{(F_\pi^{\mathrm{exp}})^2}\right]\left[1 - \frac{\Sigma_{\pi N}\,\rho_0}{(F_\pi^{\mathrm{exp}})^2 m_\pi^2}\right]^{-1},
\end{aligned}
\tag{35}
$$

where c_2 and c_3 stand for the coefficients of the effective chiral pion-nucleon Lagrangian in the heavy-baryon limit, corresponding to the terms $(v \cdot \partial\pi)^2$ and $(\partial\pi)^2$, respectively. Here, v_μ is the four velocity of the heavy baryon, whereas $\Sigma_{\pi N}$ denotes the nucleon Σ-term. The values of c_2 and c_3 are estimated for instance from the low-energy pion-nucleon scattering data, isospin-even scattering length. Note that these coefficients contain information on the p-wave contribution such as that of the Δ state [48]. Generally, the value of the ratio $(c_2 + c_3)/c_3$ is about $\frac{1}{2}$ with sign difference: $c_3 < 0$ and $c_2 > 0$ [15, 19]. Because of this sizable difference between c_2 and c_3, one observes $F_\pi^s \ll F_\pi^t$ as similar to that of the QCD sum rule calculation with the intermediate Δ-state contribution. Thus, this discrepancy from the present work may be due to the fact that the present work is only based on the quark-pion degrees of freedom.

Reference [49], in which the GOR relation was used for the time component of the axial-vector current at finite density, gives the analytic expression for the ratio as

$$
\frac{F_\pi^t(\rho_0)}{F_\pi^{\mathrm{exp}}} \approx \left[1 - \frac{\Sigma_{\pi N}\,\rho_0}{(F_\pi^{\mathrm{exp}})^2 m_\pi^2}\right]^{\frac{1}{2}},
\tag{36}
$$

9

where ρ and $\Sigma_{\pi N}$ are the nuclear matter density and πN Σ-term chosen to be 50 MeV, respectively. The corresponding result is about 0.82 at $\rho = \rho_0$.

Now, we would like to investigate the change of the pion mass at finite density. Note that in the present work we do not distinguish the pion by its charge, so that the result is charge-independent. The m_π must vanish in the chiral limit, which is the case of the present work. However, as shown in Refs. [30, 47, 50, 51], the light current-quark mass ($m_q \lesssim 5$ MeV) can easily be included within the present framework. Hence, we now extend the results for $F_\pi^{s,t}$ slightly beyond the chiral limit. For this purpose, we utilize the in-medium GOR relation [24, 25]:

$$(m_\pi^* F_\pi^t)^2 = 2m_q \langle iq^\dagger q \rangle^*, \tag{37}$$

where m_q is the current quark mass, taken to be around 5 MeV. As for the chiral condensate, we use Eq. (16) as obtained previously [33]. In the right panel of Fig. 2, we draw m_π as a function of μ. In free space, we observe $m_\pi = 139.33$ MeV which is in good agreement with the experimental value, whereas $m_\pi = 160.14$ MeV at the critical value $\mu \approx 320$ MeV. This observation tells us that the pion mass increases almost linearly and at μ_c it becomes about 15% heavier than that in free space. We summarize this result in Table I. The pion mass at finite density was also investigated within the QCD sum rule [14], meson-baryon χPT [18, 19, 20] and s-wave pion-nucleus phenomenological potential models [28, 29]. In these works, it turned out that the pion mass increases by $5 \sim 20\%$ at normal nuclear matter density which is qualitatively consistent with our result, *i.e.* it increases by about 15%. However, Ref. [17] has shown that the mass of the positive charged pion decreases as a function of nuclear density in finite nuclei or inasymmetric (neutron-dominated) nuclear matter, while the mass of the neutral pion does hardly change at all.

VI. SUMMARY AND CONCLUSION

We have investigated the pion weak decay constant and pion mass at finite density within the framework of the nonlocal chiral quark model from the instanton vacuum in the presence of the finite quark-number chemical potential. The critical value of μ was determined in the present framework consistently. The Nambu-Goldstone phase survives till $\mu = \mu_c \approx 320$ MeV, then the first-order phase transition takes place into the color-superconducting phase. The medium-modified effective chiral action being used, the pion weak decay constant was computed from the pion-to-vacuum transition matrix element. Due to the breakdown of Lorentz invariance at finite density, the time and space components of the pion weak decay constant were obtained separately. In the calculation, we assumed the soft pion.

As the final results, we obtained $F_\pi^t = 82.96$ MeV and $F_\pi^s = 80.29$ MeV at the critical value of μ. These values were in qualitative agreement with those of other theoretical models. Considering the in-medium Gell-Mann-Oakes-Renner relation, we also studied the pion mass

	F_π^s	F_π^t	m_π
$\mu = 0$	93 MeV	93 MeV	139.33 MeV
$\mu = \mu_c \approx 320$ MeV	80.29 MeV	82.96 MeV	160.14 MeV
Modification	16% ↓	13% ↓	15% ↑

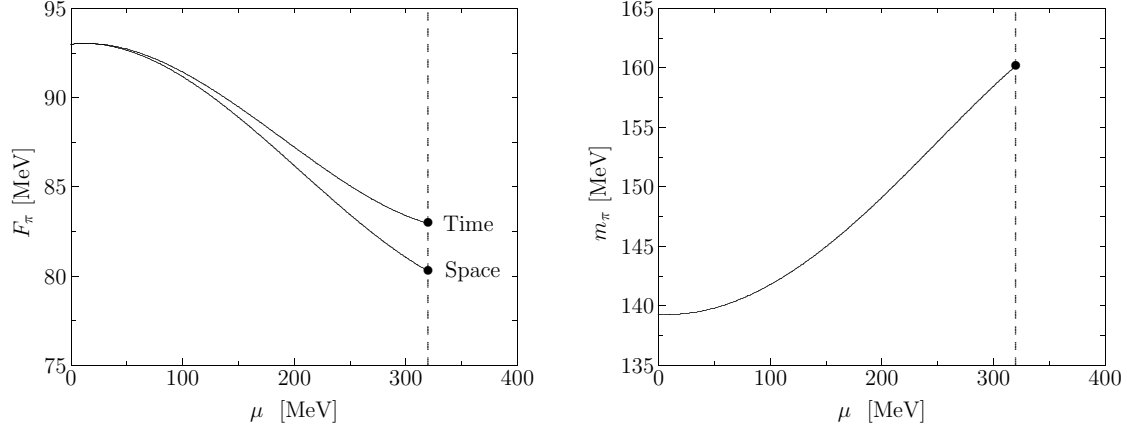TABLE I: F_π^s, F_π^t and m_π at finite density.

FIG. 2: F_π^s and F_π^t (left), and m_π (right) as functions of the quark-number chemical potential μ up to $\mu = \mu_c \approx 320$ MeV.

modification at finite density, employing the present result of the pion weak decay constant and the previously calculated chiral condensate. We found that the pion mass m_{π^+} increased almost linearly with respect to μ and at the critical value of μ it becomes about 15% larger than the free-space value: $139.33 \to 160.14$ MeV.

In general, we conclude within the present framework that the order of the medium modification for F_π and m_π is altogether about $10 \sim 20\%$ at the critical value of μ. We note that this consequence is not much different from other model estimations. However, the splitting between the time and space components of F_π has turned out to be relatively small in comparison to those in in-medium chiral perturbation theory and the QCD sum rule. As discussed in the previous sections, this difference is due to the effects of the intermediate Δ-state contribution that was not considered in this work. However, since the present results are obtained in the strict large N_c limit, we expect that the $1/N_c$ meon-loop corrections in this framework may contribute to this splitting. The corresponding investigation is under way.

Appendix

The momentum-dependent and density-dependent quark mass can be expanded as follows:

$$M(\bar{k}) \approx \mathcal{M}(k) + 2i\mu k_4 \tilde{\mathcal{M}}'(k), \quad M(\bar{k} - P) \approx \mathcal{M}(k) + 2(i\mu k_4 - k \cdot P)\tilde{\mathcal{M}}'(k),$$

$$\sqrt{M(\bar{k})} \approx \sqrt{\mathcal{M}(k)} + i\mu k_4 \frac{\tilde{\mathcal{M}}'(k)}{\sqrt{\mathcal{M}(k)}},$$

$$\sqrt{M(\bar{k} - P)} \approx \sqrt{\mathcal{M}(k)} + (i\mu k_4 - k \cdot P) \frac{\tilde{\mathcal{M}}'(k)}{\sqrt{\mathcal{M}(k)}},$$

$$\frac{1}{\sqrt{M(\bar{k})}} \approx \frac{1}{\sqrt{\mathcal{M}(k)}} \left[1 - i\mu k_4 \frac{\tilde{\mathcal{M}}'(k)}{\mathcal{M}(k)} \right],$$

$$\frac{1}{\sqrt{M(\bar{k} - P)}} \approx \frac{1}{\sqrt{\mathcal{M}(k)}} \left[1 - (i\mu k_4 - k \cdot P) \frac{\tilde{\mathcal{M}}'(k)}{\mathcal{M}(k)} \right],$$

$$\frac{\partial M(\bar{k})}{\partial k^2} \approx \tilde{\mathcal{M}}'(k) + 2i\mu k_4 \tilde{\mathcal{M}}''(k), \quad \frac{\partial M(\bar{k} - P)}{\partial k^2} \approx \tilde{\mathcal{M}}'(k) + 2(i\mu k_4 - k \cdot P)\tilde{\mathcal{M}}''(k),$$

$$\frac{1}{[\bar{k}^2 + M^2(\bar{k})][(\bar{k} - P)^2 + M^2(\bar{k})]} \approx \frac{1}{[k^2 + \mathcal{M}(k)]^2}.$$

The momentum-dependent quark mass is parameterized given below. Its derivatives with respect to the momentum are also given as follows:

$$\mathcal{M}(k) = \frac{4M_0\Lambda^4}{(k^2 + 2\Lambda^2)^2},$$

$$\tilde{\mathcal{M}}'(k) = \frac{\partial \mathcal{M}(k)}{\partial k^2} = -\frac{8M_0\Lambda^4}{(k^2 + 2\Lambda^2)^3},$$

$$\tilde{\mathcal{M}}''(k) = \frac{\partial \mathcal{M}'(k)}{\partial k^2} = \frac{24M_0\Lambda^4}{(k^2 + 2\Lambda^2)^4}.$$

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